On the Existence of Matrices with Constrained Elements, Rows and Columns

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ABSTRACT

Let the following be given: two $n\times m$ real matrices, E and F, such that $F\leqslant E$, three real n-rows, p, a and b, such that $b\leqslant a$, and three real m-columns, t, c and d, such that $d\leqslant c$. We give inequalities relating the given matrices and vectors, equivalent to the consistency of the system

$$F\leqslant X\leqslant E,\qquad d\leqslant Xt\leqslant c,\qquad b\leqslant pX\leqslant a,$$

where X is an $n\times m$ unknown real matrix.

1. RESULTS

Given a nonempty real interval A, let $a=\sup(A)\leqslant+\infty$ and $b=\inf(A)\geqslant-\infty$. We shall write $A=|a,b|$ without specifying whether or not a and b belong to A. In case $a\notin A$ (respectively, $b\notin A$) we say that a (b) is an *open extremum* of A.

Throughout this paper, I and J will stand for subsets of $N=\{1,\dots,n\}$ and $M=\{1,\dots,m\}$, respectively, and we shall write $I^c=N\setminus I$ and $J^c=M\setminus J$.

Let $p_1,\dots,p_n$, $t_1,\dots,t_m$ be $n+m$ real numbers. Given I and J as above, we define the following sets:

$$I^+=\{i\in I:p_i>0\},\qquad I^-=\{i\in I:p_i<0\},$$

$$J^+=\{j\in J:t_j>0\},\qquad J^-=\{j\in J:t_j<0\},$$

$$K=I^{c+}\times J^+\cup I^{c-}\times J^-\cup I^-\times J^{c+}\cup I^+\times J^{c-},$$

$$H=I^{c-}\times J^+\cup I^{c+}\times J^-\cup I^+\times J^{c+}\cup I^-\times J^{c-}.$$

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With this notation we state our main theorem:

THEOREM. *Let* $|b_i,a_i|$, $|d_j,c_j|$ *and* $|f_{ij},e_{ij}|$, $i=1,\dots,n$, $j=1,\dots,m$, *be* $nm+n+m$ *nonempty real intervals. Let* $p_1,\dots,p_n$, $t_1,\dots,t_m$ *be* $n+m$ *nonzero real numbers. There exists an* $n\times m$ *real matrix* $X=(x_{ij})$ *such that*

$$\sum_{j=1}^{m} x_{ij}t_j \in |b_i,a_i|, \qquad i = 1,\dots,n, \tag{1.1}$$

$$\sum_{i=1}^{n} p_i x_{ij} \in |d_j,c_j|, \qquad j = 1,\dots,m, \tag{1.2}$$

$$x_{ij} \in |f_{ij},e_{ij}|, \qquad i = 1,\dots,n, \quad j = 1,\dots,m, \tag{1.3}$$

if and only if the following relations hold:

$$\sum_K |p_it_j|e_{ij} + \sum_{I^+} |p_i|a_i + \sum_{J^-} |t_j|c_j \geqslant \sum_H |p_it_j|f_{ij} + \sum_{I^-} |p_i|b_i + \sum_{J^+} |t_j|d_j, \tag{1.4}$$

$$\sum_K |p_it_j|f_{ij} + \sum_{I^+} |p_i|b_i + \sum_{J^-} |t_j|d_j \leqslant \sum_H |p_it_j|e_{ij} + \sum_{I^-} |p_i|a_i + \sum_{J^+} |t_j|c_j, \tag{1.5}$$

for every $I\subset N$ *and* $J\subset M$. *Each inequality must be taken strict, whenever one of the* a*'s,...,*f*'s appearing in it is an open extremum of the respective interval.*

REMARK 1. The theorem may be applied to the following type of economic (transportation) problem: A factory makes n types of articles: $\alpha_1,\dots,\alpha_n$. These articles are transported to m places $\pi_1,\dots,\pi_m$ and sold there. The transportation to place π_j costs t_j per unit of article, and the article α_i is sold with a unit price p_i. A total of x_{ij} units (or units per hour) of α_i are transported and sold at π_j. One wishes to maintain under control (in a stationary regime) the following parameters: the total sales at place π_j [constraints (1.2)] and the total cost of transportation of item α_i [constraints (1.1)]. Moreover, the bipartite oriented graph of sources α_i and sinks π_j has arc capacities e_{ij} (upper capacities) and f_{ij} (lower capacities). This is expressed by (1.3).

The Theorem solves the *feasibility problem* for this system: is there any flow (x_{ij}) satisfying all the described constraints?

REMARK 2. The restrictions $p_i \neq 0$, $t_j \neq 0$ can be removed, provided we add the following to the conditions (1.4), (1.5):

$$\sum_{j \in M^-} |t_j| e_{rj} + a_r \geqslant \sum_{j \in M^+} |t_j| f_{rj}, \tag{1.6}$$

$$\sum_{j \in M^-} |t_j| f_{rj} + b_r \leqslant \sum_{j \in M^+} |t_j| e_{rj}, \tag{1.7}$$

$$\sum_{i \in N^-} |p_i| e_{is} + c_s \geqslant \sum_{i \in N^+} |p_i| f_{is}, \tag{1.8}$$

$$\sum_{i \in N^-} |p_i| f_{is} + d_s \leqslant \sum_{i \in N^+} |p_i| e_{is}, \tag{1.9}$$

$r = 1,\dots,n$, $s = 1,\dots,m$, where strict inequalities are obtained by the same rule as in the theorem.

The inequalities (1.4) and (1.5) are much simpler when the p's and t's are all positive, for in that case $K = I^c \times J$, $H = I \times J^c$ and $I^- = J^- = I^{c-} = J^{c-} = \varnothing$. An important case arises when $p_i = t_j = 1$ for all (i,j). The sums in (1.1) and (1.2) become, respectively,

the row sums of X: $R_i(X) = \sum_{j=1}^{m} x_{ij}$, and

the column sums of X: $C_j(X) = \sum_{i=1}^{n} x_{ij}$.

The following corollary is immediate:

COROLLARY 1. *Let* $|b_i, a_i|$, $|d_j, c_j|$ *and* $|f_{ij}, e_{ij}|$ *be* $nm + n + m$ *nonempty real intervals. There exists an* $n \times m$ *real matrix* $X = (x_{ij})$ *such that*

$$R_i(X) \in |b_i, a_i|, \qquad C_j(X) \in |d_j, c_j| \quad \text{and} \quad x_{ij} \in |f_{ij}, e_{ij}|,$$

for $i = 1,\dots,n$, $j = 1,\dots,m$, *if and only if*

$$\sum_{I^c \times J} e_{ij} + \sum_{I} a_i \geqslant \sum_{I \times J^c} f_{ij} + \sum_{J} d_j, \tag{1.10}$$

$$\sum_{I^c \times J} f_{ij} + \sum_{I} b_i \leqslant \sum_{I \times J^c} e_{ij} + \sum_{J} c_j \tag{1.11}$$

hold for every $I \subset N$ *and* $J \subset M$, *strict inequalities being selected by the same rule as in the theorem.*

REMARK 3. As in Remark 1, this corollary may be viewed as a supply-demand result on flows in bipartite networks (cf. [3]). A similar remark has recently been made by Moon [5], about a paper by Erdös and Minc [1].

REMARK 4. Whenever one of the a's,...,f's appearing in one of the inequalities of (1.4)–(1.11) is $\pm\infty$, the corresponding inequality is redundant. As an example, we obtain the following slight refinement of a theorem due to Ky Fan [2, Theorem 2′], generalizing results of Horn [4], and Erdös and Minc [1]:

COROLLARY 2. *Let $a_i \geqslant b_i$, $c_i \geqslant d_i$, $e_i \geqslant f_i$, $i=1,\dots,n$ be $6n$ real numbers. There exists an n-square real matrix $X=(x_{ij})$ such that*

$$b_i \leqslant R_i(X) \leqslant a_i, \qquad d_i \leqslant C_i(X) \leqslant c_i,$$

$$f_i \leqslant x_{ii} \leqslant e_i \quad \text{and} \quad x_{ij} \geqslant 0,$$

for $1 \leqslant i \neq j \leqslant n$, if and only if the following inequalities hold:

$$a_i \geqslant f_i, \quad c_i \geqslant f_i, \qquad i=1,\dots,n,$$

$$\sum_{i=1}^{n} (a_i - f_i) \geqslant \max\left(\max_{1 \leqslant i \leqslant n} (a_i + d_i - e_i - f_i), \max_{I \subset N} \left(\sum_{I} (d_i - f_i) \right) \right),$$

$$\sum_{i=1}^{n} (c_i - f_i) \geqslant \max\left(\max_{1 \leqslant i \leqslant n} (c_i + b_i - e_i - f_i), \max_{I \subset N} \left(\sum_{I} (b_i - f_i) \right) \right).$$

2. PROOFS

We shall prove the theorem and Remark 2 simultaneously. Thus, assume that $p_1,\dots,p_n$, $t_1,\dots,t_m$ are arbitrary real numbers. We adopt here some of the concepts and results of [6, pp. 203–206]. The system of inequalities (1.1)–(1.3) can be rewritten as

$$\sum_{j=1}^{m} x_{ij} t_j = y_i, \tag{3.1}$$

$$\sum_{i=1}^{n} p_i x_{ij} = z_j, \qquad i=1,\dots,n, \quad j=1,\dots,m, \tag{3.2}$$

$$y_i \in |b_i, a_i|, \tag{3.3}$$

$$z_j \in |d_j, c_j|, \tag{3.4}$$

$$x_{ij} \in |f_{ij}, e_{ij}|. \tag{3.5}$$

If we introduce the nm-column x, whose coordinates consist of the entries of X ordered lexicographically, then (3.1) and (3.2) can be shortened to

$$Ax = y \oplus z, \tag{3.6}$$

where $y \oplus z = (y_1, \dots, y_n, z_1, \dots, z_m)^t$, and A is the $(n+m) \times nm$ matrix

$$A = \begin{bmatrix} t & 0 & \dots & 0 & D_1 \\ 0 & t & \dots & 0 & D_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & t & D_n \end{bmatrix}^T. \tag{3.7}$$

Here $t = (t_1, \dots, t_m)^T$, $D_i = p_i I_m$ ($I_m = m$-identity-matrix), $i = 1, \dots, n$, and the 0's are zero m-columns.

The vector space $L^{\perp} \subset \mathbb{R}^{nm+n+m}$ orthogonal to $L = \{x \oplus y \oplus z : Ax = y \oplus z\}$ is given by

$$L^{\perp} = \{x^* \oplus y^* \oplus z^* : x^* = -A^T(y^* \oplus z^*)\}. \tag{3.8}$$

Define the *support* of $w \in \mathbb{R}^q$ as the set

$$\operatorname{supp}(w) = \{k : 1 \leqslant k \leqslant q,\ w_k \neq 0\}.$$

To apply Theorem 22.6 of [6], we need a lemma on the *elementary vectors* of $L^{\perp}$, i.e., the nonzero vectors of $L^{\perp}$ with minimal support.

LEMMA. *An elementary vector of $L^{\perp}$ is a multiple of a vector ω^* of one of the following types*:

Type 1. $\omega^* = \chi^* \oplus \eta^* \oplus \zeta^* \in L^{\perp}, \# \operatorname{supp}(\eta^* \oplus \zeta^*) = 1$, *and the nonzero coordinate of $\eta^* \oplus \zeta^*$ equals* 1;

Type 2. $\omega^* = \chi^* \oplus \eta^* \oplus \zeta^* \in L^{\perp}$, *and for some $I \subset N$ and $J \subset M$, we have $\eta_i^* = p_i$ for $i \in I$, $\eta_i^* = 0$ for $i \in I^c$, $\zeta_j^* = -t_j$ for $j \in J$, and $\zeta_j^* = 0$ for $j \in J^c$.*

Proof of the Lemma. Let $v^* = x^* \oplus y^* \oplus z^*$ be an elementary vector of $L^\perp$, which is not a multiple of a vector of Type 1. Let $I = \operatorname{supp}(y^*)$ and $J = \operatorname{supp}(z^*)$. If $I = \varnothing$, then there exist r and s in J, $r \neq s$. The vector of type 1, such that $\zeta_r^* = 1$, has its support strictly contained in $\operatorname{supp}(v^*)$, which is absurd. Therefore $I \neq \varnothing$. Analogously $J \neq \varnothing$. From (3.8), and the form (3.7) of the matrix A, it is easily seen that the coordinates of x^* are

$$x_{ij}^* = -y_i^* t_j - p_i z_j^*, \qquad \text{for} \quad (i,j) \in I \times J, \tag{3.9}$$

$$x_{ij}^* = -y_i^* t_j, \qquad \text{for} \quad (i,j) \in I \times J^c, \tag{3.10}$$

$$x_{ij}^* = -p_i z_j^*, \qquad \text{for} \quad (i,j) \in I^c \times J, \tag{3.11}$$

$$x_{ij}^* = 0, \qquad \text{for} \quad (i,j) \in I^c \times J^c. \tag{3.12}$$

Now, if $i_0 \in I$, then $p_{i_0} \neq 0$. Otherwise, the vector $\bar{x}^*$, obtained from (3.9)–(3.12) by letting $y_{i_0}^* = 0$, satisfies $\operatorname{supp}(\bar{x}^*) \subset \operatorname{supp}(x^*)$, and we could construct a nonzero $\bar{v}^* \in L^\perp$, the support of which was strictly contained in $\operatorname{supp}(v^*)$. Analogously, we have $t_j \neq 0$ for $j \in J$.

Whether or not the coordinates (3.10)–(3.11) are zero is independent of the values of $y_i^* \neq 0$, or of $z_j^* \neq 0$. Moreover, the coordinates (3.9) are zero only if $z_j^*/t_j = -y_i^*/p_i$. The lemma follows. ∎

REMARK 5. We could characterize completely the elementary vectors of $L^\perp$ as being either nonzero multiples of vectors of type 1, or such that $y_i^* = cp_i$, $z_j^* = -ct_j$ for all $(i,j) \in \operatorname{supp}(y^*) \times \operatorname{supp}(z^*) \neq \varnothing$, for some $c \neq 0$. Actually, we do not need so much.

An easy consequence of [6, Theorem 22.6] is that the system (3.1)–(3.5) is consistent if and only if every vector ω^* of type 1, or of type 2, satisfies the condition

$$0 \in A(\omega^*), \tag{3.13}$$

where $A(\omega^*)$ is the interval

$$A(\omega^*) = \sum_{i=1}^{n} \sum_{j=1}^{m} \chi_{ij}^* |f_{ij}, e_{ij}| + \sum_{i=1}^{n} \eta_i^* |b_i, a_i| + \sum_{j=1}^{m} \zeta_j^* |d_j, c_j|. \tag{3.14}$$

Equation (3.13) is equivalent to the following relations:

$$\sup(A(\omega^*)) \geqslant 0, \tag{3.15}$$

$$\inf(A(\omega^*)) \leqslant 0, \tag{3.16}$$

where the inequality (3.15) [respectively (3.16)] is strict, whenever one of the $nm+n+m$ intervals, summed on the right side of (3.14), is open on the right [on the left].

Taking, in (3.15)–(3.16) successively, ω^* equal to its two possible types given by the lemma, we obtain the six sets of inequalities (1.4)–(1.9). We work out the case (3.15), when ω^* is a vector of Type 2. We then have

$$\begin{aligned}\sup(A(\omega^*)) &= \sup\Big(\sum_{I^c\times J} p_i t_j |f_{ij}, e_{ij}| - \sum_{I\times J^c} p_i t_j |f_{ij}, e_{ij}| \\ &\quad + \sum_I p_i |b_i, a_i| - \sum_J t_j |d_j, c_j| \Big) \\ &= \sup\Big(\sum_K |p_i t_j| . |f_{ij}, e_{ij}| - \sum_H |p_i t_j| \cdot |f_{ij}, e_{ij}| + \sum_{I^+} |p_i| \cdot |b_i, a_i| \\ &\quad - \sum_{I^-} |p_i| \cdot |b_i, a_i| - \sum_{J^+} |t_j| \cdot |d_j, c_j| + \sum_{J^-} |t_j| . |d_j, c_j| \Big) \\ &= \sum_K |p_i t_j| e_{ij} \\ &\quad - \sum_H |p_i t_j| f_{ij} + \sum_{I^+} |p_i| a_i \\ &\quad - \sum_{I-} |p_i| b_i - \sum_{J^+} |t_j| d_j + \sum_{J^-} |t_j| c_j .\end{aligned}$$

The set of inequalities (1.4) follows. The others [(1.5)–(1.9)] are obtained by similar calculations. That part of the theorem selecting the strict inequalities is an easy consequence of the observations following (3.16).

In the case when $p_i \neq 0$, $t_j \neq 0$ for all (i,j), all the elementary vectors are multiples of vectors of type 2. Therefore, (1.6)–(1.9) are, in that case, consequences of (1.4)–(1.5). The proof is now complete. ■

Proof of Corollary 2. Apply Corollary 1 to the present situation, taking $m=n$, $f_i = f_{ii}$, $e_i = e_{ii}$, $f_{ij}=0$ and $e_{ij} = +\infty$, for $i \neq j$. By Remark 4, we have only to take into account those inequalities in (1.10)–(1.11) corresponding to the following sets, I and J:

in (1.10):	in (1.11):
$I^c \times J = \{(i,i)\}$, $i \in N$,	$I \times J^c = \{(i,i)\}$, $i \in N$,
$I^c = \varnothing$ and $J \subset N$,	$J^c = \varnothing$ and $I \subset N$,
$J = \varnothing$ and $\# I = 1$,	$I = \varnothing$ and $\# J = 1$.

Observe that we may only consider those inequalities coming from elementary vectors of $L^{\perp}$. Consequently, when we put $J=\varnothing$ (respectively $I=\varnothing$), we may restrict ourselves to the cases when $\#I=1$ (when $\#J=1$). Corollary 2 follows after some easy computations. ■

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REFERENCES

1 Paul Erdös and Henrik Minc, Diagonals of nonnegative matrices, *Linear and Multilinear Algebra* 1:89–95 (1973).
2 Ky Fan, Two applications of a consistency theorem for systems of linear inequalities, *Linear Algebra Appl.* 11:171–180 (1975).
3 L. R. Ford and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, New Jersey, 1962, Chapter 2, Sec. 2.
4 A. Horn, Doubly stochastic matrices and the diagonal of a rotation matrix, *Amer. J. Math.* 76:620–630 (1954).
5 J. W. Moon, A note on matrices with given diagonal entries, *Linear and Multilinear Algebra* 4:177–178 (1976).
6 R. T. Rockafellar, *Convex Analysis*, Princeton U. P., Princeton, N. J., 1970.

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